Problem. (1.2) Let \( S = \{-2, -1, 0, 1, 2, 3\} \). Describe each of the following sets as \( \{x \in S : p(x)\} \), where \( p(x) \) is some condition on \( x \).

(a) \( A = \{1, 2, 3\} \)

(b) \( B = \{0, 1, 2, 3\} \)

(c) \( C = \{-2, -1\} \)

(d) \( D = \{-2, 2, 3\} \)

Solution. The sets may be described as:

(a) \( A = \{x \in S : x > 0\} \),

(b) \( B = \{x \in S : x > -1\} \),

(c) \( C = \{x \in S : x < 0\} \), and

(d) \( D = \{x \in S : |x| > 1\} \).

\[ \square \]

Remark. There are many ways to describe these sets. The main thing we are looking for is that the elements come from appropriate sets. For example, I wrote “\( x \in S \)”. Some of you chose to use “\( x \in \mathbb{Z} \)” or some similarly appropriate set. This is fine, so long as the condition \( p(x) \) holds. For example, set \( D \) can be written as

\[ D = \{x \in \mathbb{Z} : (x + 2)(x - 2)(x - 3) = 0\}. \]

While mathematically correct, in this case we prefer to use “\( x \in S \)” given the context of the problem.
Problem. (1.4) Write each of the following sets by listing its elements with braces.

(a) \( A = \{ n \in \mathbb{Z} : -4 < n \leq 4 \} \)

(b) \( B = \{ n \in \mathbb{Z} : n^2 < 5 \} \)

(c) \( C = \{ n \in \mathbb{N} : n^3 < 100 \} \)

(d) \( D = \{ x \in \mathbb{R} : x^2 - x = 0 \} \)

(e) \( E = \{ x \in \mathbb{R} : x^2 + 1 = 0 \} \)

Solution. The sets written explicitly are:

(a) \( A = \{-3, -2, -1, 0, 1, 2, 3, 4\} \),

(b) \( B = \{-2, -1, 0, 1, 2\} \),

(c) \( C = \{1, 2, 3, 4\} \),

(d) \( D = \{0, 1\} \), and

(e) \( E = \emptyset \).

Remark. There are several issues to resolve in this problem.

(a) According to the textbook, \( \mathbb{N} \) does not include the number 0. While many mathematicians disagree whether 0 is included, we will stick to the definition in the book.

(b) When asked to list the elements, we avoid using an ellipsis. If an ellipsis is required, though, use \( \cdots \); similar commands: \( \cdots \) and \( \dotsc \). We avoid explicitly typing “\ldots” and use the appropriate commands instead.
Problem. (1.8) Let

\[ A = \{ n \in \mathbb{Z} : 2 \leq |n| < 4 \}, \]
\[ B = \{ x \in \mathbb{Q} : 2 < x \leq 4 \}, \]
\[ C = \{ x \in \mathbb{R} : x^2 - (2 + \sqrt{2})x + 2\sqrt{2} = 0 \}, \text{ and} \]
\[ D = \{ x \in \mathbb{Q} : x^2 - (2 + \sqrt{2})x + 2\sqrt{2} = 0 \}. \]

(a) Describe the set \( A \) by listing its elements.

(b) Give an example of three elements that belong to \( B \) but not to \( A \).

(c) Describe the set \( C \) by listing its elements.

(d) Describe the set \( D \) in another manner.

(e) Determine the cardinality of each of the sets \( A \), \( C \) and \( D \).

Solution.

(a) The set \( A \) listed explicitly is \( A = \{-3, -2, 2, 3\} \).

(b) Since the elements in \( B \) are rational numbers, any three rational numbers (that are not simultaneously integers) in \( B \) will not be in \( A \). Three such examples are: \( \frac{5}{2}, \frac{7}{2}, \) and \( \frac{13}{4} \).

(c) Note that \( x^2 - (2 + \sqrt{2})x + 2\sqrt{2} = 0 \) can be written as \((x - 2)(x - \sqrt{2}) = 0\). This implies that \( C = \{2, \sqrt{2}\} \).

(d) Since \( D \) is restricted to \( \mathbb{Q} \), we have \( D = \{2\} \).

(e) The cardinalities are: \( |A| = 4, |C| = 2, \) and \( |D| = 1 \).

Remark. There are infinite possibilities for part (b). For part (c), we could have optionally used the quadratic formula if the factorization is not readily recognized. As for set \( D \), note that \( \mathbb{Z} \subset \mathbb{Q} \), so it is not necessary to write 2 as a fraction.
**Problem.** (1.10) Give examples of three sets $A$, $B$ and $C$ such that

(a) $A \subseteq B \subset C$

(b) $A \in B$, $B \in C$, and $A \notin C$

(c) $A \in B$ and $A \subset C$.

**Solution.**

(a) If $A = \{1\}$, $B = \{1\}$, and $C = \{1, 2\}$, then $A \subseteq B$ since they are the same sets. And $B \subset C$ since the element of $B$ is also an element of $C$.

(b) Let $A = \{1\}$, $B = \{\{1\}\}$, and $C = \{\{\{1\}\}\}$. We can easily check that $A \in B$, and $B \in C$. Lastly, $A \notin C$ since $C$ contains only one element, namely the set $B$.

(c) Suppose $A = \{1\}$, $B = \{\{1\}\}$, and $C = \{1, 2\}$. It follows that $A \in B$ and $A \subset C$ as all elements of $A$ are also elements of $C$.

**Remark.** This problem requires more than just simply listing possible sets $A$, $B$, and $C$. We must not only ensure that our choices for $A$, $B$, and $C$ are sets, but also explain why the conditions are satisfied. Interestingly, the authors neglected to use proper punctuation to separate their cases in the problem statement.
Problem. (1.14) Find $\mathcal{P}(A)$ and $|\mathcal{P}(A)|$ for

(a) $A = \{1, 2\}$.

(b) $A = \{\emptyset, 1, \{a\}\}$.

Solution. The power set of the given sets and the respective cardinality are:

(a) $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ and $|\mathcal{P}(A)| = 2^2 = 4$; and

(b) $\mathcal{P}(A) = \{\emptyset, \emptyset, \{1\}, \{\{a\}\}, \{\emptyset, 1\}, \{\emptyset, \{a\}\}, \{1, \{a\}\}, \{\emptyset, 1, \{a\}\}\}$ and $|\mathcal{P}(A)| = 2^3 = 8$. 

□
Problem. (1.18) For $A = \{ x : x = 0 \text{ or } x \in \mathcal{P}(\{0\}) \}$, determine $\mathcal{P}(A)$.

Solution. First note that $\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$. For the sake of clarity, write $\mathcal{P}(\{\emptyset\}) = \{x_1, x_2\}$. Then $A = \{0, x_1, x_2\}$. To determine $\mathcal{P}(A)$, consider all possible subsets of $A$ and then replace $x_1$ and $x_2$ with their actual representation. The possible subsets of $A$ are

$\emptyset, \{0\}, \{x_1\}, \{x_2\}, \{0, x_1\}, \{0, x_2\}, \{x_1, x_2\}, \text{ and } \{0, x_1, x_2\}.$

Therefore

$$\mathcal{P}(A) = \{\emptyset, \{0\}, \{\emptyset\}, \{\{\emptyset\}\}, \{0, \emptyset\}, \{0, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}, \{0, \emptyset, \{\emptyset\}\}\}.$$ 

Remark. The “or” in the definition of $A$ does not mean that $A$ has two possible representations. The “or” is part of the condition $p(x)$ when using the notation $\{ x \in S : p(x) \}$. That is $x = 0$ or $x$ is one of the elements of $\mathcal{P}(\{\emptyset\})$. Notice how use of substitution makes the task of determining the subsets of $A$ much easier. No longer do we not get lost in all the ”s and ’s, nor is there confusion as to whether to write $\emptyset$ or $\{\emptyset\}$ (a set whose only member is the empty set), or even $\{\{\emptyset\}\}$. Lastly, when typing up sets contained within sets, consider separating parts into blocks to help you separate the members of the power set. As an example, you might consider listing all one-element subsets in one row, and all two-element sets in the next row, etc.
Problem. (1.21) Three subsets $A$, $B$ and $C$ of $\{1,2,3,4,5\}$ have the same cardinality. Furthermore,

(a) 1 belongs to $A$ and $B$ but not $C$.
(b) 2 belongs to $A$ and $C$ but not $B$.
(c) 3 belongs to $A$ and exactly one of $B$ and $C$.
(d) 4 belongs to an even number of $A$, $B$ and $C$.
(e) 5 belongs to an odd number of $A$, $B$ and $C$.
(f) The sums of the elements in two of the sets $A$, $B$ and $C$ differ by 1.

What is $B$?

Solution. Conditions (a), (b), and (c) imply $\{1,2,3\} \subseteq A$. This suggests that $|A|$ is 3, 4, or 5. Since $|A| = |B| = |C|$, and we know that $1 \notin B$, the cardinality of $B$ (and hence of $A$ and $C$) cannot be 5. If $|A| = |B| = |C| = 4$, then (a) and (b) imply $B = \{1,3,4,5\}$ and $C = \{2,3,4,5\}$. However, this result is impossible since it contradicts (c). So we must have $|A| = |B| = |C| = 3$. Therefore $A = \{1,2,3\}$, and $B$ and $C$ must each be sets consisting of three elements. We also now know that 4 and 5 are not elements of $A$. Consequently, (e) implies 5 must belong either $B$ or $C$, but not both. And (d) implies 4 must either belong to both $B$ and $C$, or 4 is not an element of any set. At this point, we have enough information to conclude

$$A = \{1,2,3\}, \quad B = \{1,b_1,b_2\}, \quad \text{and} \quad C = \{2,c_1,c_2\}.$$ 

The possible values for $b_1$, $b_2$, $c_1$, and $c_2$ are 3, 4, and 5. If $4 \notin B$ and $4 \notin C$, then one or both of $B$ and $C$ will not have three elements. Therefore 4 must belong to both $B$ and $C$. Lastly, condition (f) forces $5 \in B$ and $3 \in C$. Our three sets are

$$A = \{1,2,3\}, \quad B = \{1,4,5\}, \quad \text{and} \quad C = \{2,3,4\}.$$ 

Remark. A complete solution must also justify why $B$ satisfies all the given conditions. Implicitly, we would have to then also figure out $A$ and $C$ in order to verify that what we choose for the subset $B$ is in fact the correct answer.
Problem. (1.22) Let $U = \{1, 3, \ldots, 15\}$ be the universal set, $A = \{1, 5, 9, 13\}$, and $B = \{3, 9, 15\}$. Determine the following:

(a) $A \cup B$
(b) $A \cap B$
(c) $A - B$
(d) $B - A$
(e) $\overline{A}$
(f) $A \cap \overline{B}$.

Solution. The sets are:

(a) $A \cup B = \{1, 3, 5, 9, 13, 15\}$,
(b) $A \cap B = \{9\}$,
(c) $A - B = \{1, 5, 13\}$,
(d) $B - A = \{3, 15\}$,
(e) $\overline{A} = \{3, 7, 11, 15\}$, and
(f) $A \cap \overline{B} = \{1, 5, 13\}$. 

\qed
Problem. (1.26) Let $U$ be a universal set and let $A$ and $B$ be two subsets of $U$. Draw a Venn diagram for each of the following sets.

(a) $A \cup B$
(b) $A \cap B$
(c) $A \cap B$
(d) $A \cup B$

What can you say about parts (a) and (b)? parts (c) and (d)?

Solution.

(a) First we determine $A \cup B$.

\begin{center}
\begin{tikzpicture}
  \node at (0,0) {\includegraphics[width=0.5\textwidth]{venn_diagram}};
  \node at (0,2) {$U$};
  \node at (-1,0) {$A$};
  \node at (1,0) {$B$};
\end{tikzpicture}
\end{center}

It follows that $\overline{A \cup B}$ is the complement of the union above, and the Venn diagram is given below.

\begin{center}
\begin{tikzpicture}
  \node at (0,0) {\includegraphics[width=0.5\textwidth]{venn_diagram}};
  \node at (0,2) {$U$};
  \node at (-1,0) {$A$};
  \node at (1,0) {$B$};
\end{tikzpicture}
\end{center}

(b) Below are the Venn diagrams for $\overline{A}$ and $\overline{B}$.

\begin{center}
\begin{tikzpicture}
  \node at (0,0) {\includegraphics[width=0.5\textwidth]{venn_diagram}};
  \node at (0,2) {$U$};
  \node at (-1,0) {$A$};
  \node at (1,0) {$B$};
\end{tikzpicture}
\end{center}

(continued on next page)
The intersection $\overline{A} \cap \overline{B}$ is shown below.

(c) Note that $A \cap B$ has the Venn diagram below.

Therefore the following diagram is a Venn diagram for $A \cap B$.

(d) Using what we have in part (b), it is evident that $\overline{A} \cup \overline{B}$ has the following Venn diagram.

Based on our Venn diagrams, $\overline{A} \cup \overline{B} = \overline{A \cap B}$ and $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Remark. In a sense, the results of this problem give us a “distribution” rule for complements over unions and intersections.
Problem. (1.32) Give an example of four different subsets $A$, $B$, $C$ and $D$ of \{1, 2, 3, 4\} such that all intersections of two subsets are different.

Solution. One possible combination of subsets is:

(a) $A = \{1, 2, 3\}$,
(b) $B = \{2, 3, 4\}$,
(c) $C = \{1, 3, 4\}$, and
(d) $D = \{1, 4\}$.

We can easily verify that no two intersections are the same.

\[
\begin{align*}
A \cap B &= \{2, 3\}, & A \cap C &= \{1, 3\}, & A \cap D &= \{1\} \\
B \cap C &= \{3, 4\}, & B \cap D &= \{4\}, & C \cap D &= \{1, 4\}
\end{align*}
\]